Sufficiency of the Ricci equations for characterizing the Riemann tensor

S. BRIAN EDGAR

Department of Mathematics
Linköping University
S - 581 83 Linköping, Sweden

Abstract. *It is shown – in very general circumstances, for a space-time with Lorentz metric* g *– that a fourth order tensor with all the algebraic symmetries of a Riemann tensor and which satisfies the Ricci equations (with covariant derivative constructed from* g *in the usual way) is of necessity the Riemann tensor of the metric* g. *Rendall's conjecture that this result holds in the generic case is confirmed, and his counterexamples are shown to be part of a small set of very specialised spaces which do not obey the general rule.*

1. INTRODUCTION

For a 4-dimensional manifold on which the covariant derivative is defined in terms of a Lorentz metric g_{ab}, it has been investigated, [1] whether a tensor (with the appropriate algebraic symmetries) satisfying the Bianchi equations is of necessity the Riemann tensor associated with that metric g_{ab}. Although there are non-trivial classes of spaces where such a tensor is identical to the Riemann tensor, as well as other special classes where the tensor is «close to» the Riemann tensor, it is clear, for more general classes of spaces, that the Bianchi equations alone are not sufficient to guarantee a Riemann tensor.

Rendall [2, 3] has recently discussed whether «curvature candidates» (tensors with the appropriate symmetry properties) which obey the Ricci equations as well as the Bianchi equations are of necessity Riemann tensors. He has found some very special examples where such a curvature candidate is *not* a Riemann tensor – even of some metric other than that associated with the covariant derivative used in the Bianchi and Ricci

Key-Words: Riemann tensor, Ricci equations, integrability conditions.
1980 MSC: 53 B 20, 58 G 99.

equations [2]. In fact, Rendall suspected that his counterexamples might be pathological ones, and in [3] sketched the steps of a proof which would give an affirmative answer to his question in the «generic» case i.e. for an open dense set of curvature candidates in the Whitney C^∞ topology; the argument in his proof depends on the validity of a certain uproven algebraic condition at a point.

In this paper we shall look at the situation where the curvature candidate satisfies the Ricci equations, and we consider the question whether a curvature candidate satisfying the Ricci equations is the Riemann tensor of the metric associated with the covariant derivative used in the Ricci equations. In a subsequent paper we shall consider the more general question as to whether such a curvature candidate can be the Riemann tensor of some other metric.

Consider a curvature candidate K^a_{bcd} satisfying,

(1a) $$K^a_{bcd} = -K^a_{bdc}$$

(1b) $$K^a_{[bcd]} = 0$$

(1c) $$g_{ai}K^i_{bcd} = -g_{bi}K^i_{acd}$$

and satisfying the Ricci equations,

(2) $$2\,K^a_{bcd;[ef]} = -K^i_{bcd}K^a_{ief} + K^a_{icd}K^i_{bef} + K^a_{bic}K^i_{cef} + K^a_{bic}K^i_{def}$$

where the covariant derivative is defined in terms of the Lorentz metric g_{ab}, which is used to raise and lower indices in the usual way. We wish to investigate when the curvature candidate K^a_{bcd} is equal to the Riemann tensor R^a_{bcd} of the metric g_{ab}.

Equations (1c) and (2) impose constraints on both g_{ab} and K^a_{bcd} and so for *arbitrary* K^a_{bcd} (with the symmetries (1a,b)) there is no guarantee that there exists a non-degenerate metric tensor g_{ab} satisfying (1c) and (2). We will therefore be considering only the class of curvature candidates K^a_{bcd} for which there exists a non-degenerate metric tensor g_{ab} which satisfies (1c) and (2) (if the metric tensor satisfying (1c) is permitted to differ from that satisfying (2), then there would appear to be a larger class of curvature candidates K^a_{bcd} to be considered).

In Section 2 it is shown that equations (2), together with the usual definition of the Riemann tensor, lead to an algebraic constraint, which can be split into two independent sets of equations. These constraint equations are used directly in Section 3 to show – in very general circumstances – that any tensor K^a_{bcd} satisfying the conditions (1) and (2) is equal to the Riemann tensor R^a_{bcd} associated with the metric g_{ab} in the usual way.

In Section 4 some questions and developments arising from this result are considered. It is shown that the counterexamples given by Rendall [2] do not belong to the class of curvature candidates for which our theorem is valid, and so do not contradict our result. In addition our result shows that Rendall's uproven algebraic condition does in fact hold, for the class of curvature candidates satisfying (1) and (2), and so it is confirmed that a generic curvature candidate satisfying (1) and (2) is equal to the Riemann tensor of the metric g_{ab}. Finally, the answer as to whether – for the set of curvature candidates K^a_{bcd} satisfying (1) and (2) – K^a_{bcd} can be the Riemann tensor of any other metric than g_{ab}, is shown to be negative, in very general circumstances; but the class of spaces for which this uniqueness result holds will be determined explicitly in the subsequent paper when the second part of Rendall's question is considered.

As noted above, Rendall [2, 3] considered the additional constraint of the Bianchi equation on K^a_{bcd},

(3) $$K^a_{b[cd;e]} = 0 .$$

However the result proved here requires only the Ricci constraint (2) on K^a_{bcd}.

2. PRELIMINARY CALCULATIONS

All tensors, of course, satisfy the usual Ricci equation which, in the case of K^a_{bcd}, is

(4) $$2\,K^a_{bcd;[ef]} = -K^i_{bcd}R^a_{ief} + K^a_{icd}R^i_{bef} + K^a_{bid}R^i_{cef} + K^a_{bci}R^i_{def}$$

Subtracting (4) from (3) gives the algebraic constraint equation,

(5) $$0 = K^i_{bcd}P^a_{ief} - K^a_{icd}P^i_{bef} - K^a_{bid}P^i_{cef} - K^a_{bci}P^i_{def}$$

where

(6) $$P^a_{bcd} = K^a_{bcd} - R^a_{bcd} .$$

The condition to be proven is that equations (5) imply that P^a_{bcd} must be identically zero. As Rendall has pointed out, [3], the set of equations (5) is a very large set in a large number of variables; however it can be divided into more manageable subsets by decomposing both R^a_{bcd} and K^a_{bcd} (and hence P^a_{bcd}).

Since K^a_{bcd} has exactly the same algebraic symmetries (1) as the Riemann tensor R^a_{bcd} it will be convenient to decompose it in the same way as the Riemann tensor is decomposed,

(7) $$K_{abcd} = \underset{0}{K}_{abcd} + \frac{1}{2}\left(g_{ac}\underset{0}{K}_{bd} + g_{bd}\underset{0}{K}_{ac} - g_{ad}\underset{0}{K}_{bc} - g_{bc}\underset{0}{K}_{ad}\right) + \frac{K}{12}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) .$$

Equations (5) can be split into one subset containing only $\underset{0}{K}_{ab}$

$$0 = \underset{0}{K}_{ic}\ P^{i}_{bef} + \underset{0}{K}_{ib}\ P^{i}_{cef} \tag{8}$$

and the other subset contains only $\underset{0}{K}^{a}_{bcd}$

$$0 = \underset{0}{K}^{i}_{bcd}\ P^{a}_{ief} - \underset{0}{K}^{a}_{icd}\ P^{i}_{bef} - \underset{0}{K}^{a}_{bid}\ P^{i}_{cef} - \underset{0}{K}^{a}_{bci}\ P^{i}_{def} \tag{9}$$

By decomposing P^{a}_{bcd} in the same way, equation (8) can itself be subdivided into a pair of equations,

$$\begin{aligned} 0 = {} & \underset{0}{K}^{j}_{i}\ \underset{0}{P}^{i}_{cej} + \frac{3}{2}\underset{0}{K}^{j}_{c}\ \underset{0}{P}_{ie} + \frac{1}{2}\underset{0}{K}^{i}_{e}\ \underset{0}{P}_{ic} \\ & + \frac{1}{3}P\underset{0}{K}_{ec} - \frac{1}{2}\eta_{ec}\underset{0}{P}_{ij}\ \underset{0}{K}^{ij} \end{aligned} \tag{8a}$$

$$\begin{aligned} 0 = {} & \underset{0}{K}_{ic}\ \underset{0}{P}^{i}_{bef} + \underset{0}{K}_{ib}\ \underset{0}{P}^{i}_{cef} + \frac{1}{2}\eta_{b[e}\ \underset{0}{K}^{j}_{|i}\ \underset{0}{P}^{i}_{c]f|j} \\ & + \frac{1}{2}\eta_{c[e}\ \underset{0}{K}^{j}_{|i}\ \underset{0}{P}^{i}_{b]f|j} + \underset{0}{K}_{c[e}\underset{0}{P}_{f]b} + \underset{0}{K}_{b[e}\underset{0}{P}_{f]c} \end{aligned} \tag{8b}$$

Two different set of constraints are therefore imposed independently on P^{a}_{bcd} – through (8) and (9) – by different parts of K^{a}_{bcd}. To examine these in detail it will be convenient to classify $\underset{0}{K}^{a}_{bcd}$ and $\underset{0}{K}_{ab}$ and to consider the canonical forms of the different classes. Since $\underset{0}{K}^{a}_{bcd}$ has all the algebraic symmetries of the Riemann tensor as well as being trace-free, it can be classified by the Petrov scheme [6,7]. The Plebanski classification is based on the Plebanski tensor – a tensor with all the algebraic symmetries of the trace-free Riemann tensor – which can be constructed from any symmetric second order trace-free tensor [6]. The Plebanski tensor and associated classification scheme have been written in N.P. notation [5] and have already proved useful in investigating algebraic constraints on the Riemann tensor [8]. The Plebanski tensor $\mathcal{K}^{ab}_{cd}$ associated with the trace-free second order symmetric tensor $\underset{0}{K}_{ab}$ is defined by

$$\mathcal{K}^{ab}{}_{cd} = \underset{0}{K}{}^{[a}{}_{[c}\underset{0}{K}{}^{b]}{}_{d]} + \delta^{[a}{}_{[c}\underset{0}{K}{}_{d]i}\ \underset{0}{K}{}^{b]i} - \frac{1}{6}\delta^{[a}{}_{[c}\delta^{b]}{}_{d]}\underset{0}{K}{}^{ij}\ \underset{0}{K}{}_{ij} \tag{10}$$

and the Plebanski tensor $\mathcal{P}^{ab}{}_{cd}$ associated with the tensor $\underset{0}{P}_{ab}$ is defined in the same way.

3. SUFFICIENCY

When the N.P. version of the canonical forms of the different Petrov types of $\underset{0}{K}_{abcd}$ are substituted into (9) it is found for Petrov types I, II, III that P_{abcd} is identically zero. However for Petrov types D, N there are some components of P_{abcd} that are not identically zero; in each of these cases we can classify the non-zero P_{abcd} according to its trace-free part $\underset{0}{P}_{abcd}$ and its Plebanski part $\mathcal{P}_{abcd}$, using Table 1 in reference [7]. This classification gives a description of the non-zero P_{abcd} independent of the tetrad choice which was used to put P_{abcd} into canonical form. The results are illustrated in the Appendix and can be summarised as follows,

(i) If $\underset{0}{K}_{abcd}$ is Petrov type I, II or III then the tensor P_{abcd} is identically zero.

(ii) If $\underset{0}{K}_{abcd}$ is Petrov type D then *either* P_{abcd} is identically zero, *or* at least one of $\underset{0}{P}_{abcd}$ and $\mathcal{P}_{abcd}$ is Petrov type D.

(iii) If $\underset{0}{K}_{abcd}$ is Petrov type N then *either* P_{abcd} is identically zero, *or* $\underset{0}{P}_{abcd}$ is type N (or O) and $\mathcal{P}_{abcd}$ is Petrov type O.

When the N.P. versions of the canonical forms of the different Petrov types of the Plebanski tensor $\mathcal{K}_{abcd}$ [6,8] are substituted into equations (8a,b) a similar pattern is found as above. Once again, for those classes where P_{abcd} is not identically zero, we can find its classes. The results, also illustrated in the Appendix, are summarised as follows,

(i) If $\mathcal{K}_{abcd}$ is Petrov type I, II or III then the tensor P_{abcd} is identically zero.

(ii) If $\mathcal{K}_{abcd}$ is Petrov type D then *either* P_{abcd} is identically zero, *or* at least one of $\underset{0}{P}_{abcd}$ and $\mathcal{P}_{abcd}$ is Petrov type D.

(iii) If $\mathcal{K}_{abcd}$ is Petrov type N *either* P_{abcd} is identically zero, *or* $\underset{0}{P}_{abcd}$ is type N and $\mathcal{P}_{abcd}$ is Petrov type O.

We now combine these results. Since the canonical form for a particular Petrov class of $\underset{0}{K}_{abcd}$ is obtained by a different tetrad choice than the canonical form for a particular Petrov class of $\mathcal{K}_{abcd}$ we cannot compare directly the non-zero components of P_{abcd} obtained in the two situations; but rather the invariant classes of the non-zero P_{abcd} can be compared. In this way it is easy to see for example, that if $\underset{0}{K}_{abcd}$ is Petrov type N and $\mathcal{K}_{abcd}$ is Petrov type D, then P_{abcd} must be identically zero. The combined results are given below.

A curvature candidate K^a_{bcd} satisfying (1a, b, c) and which also satisfies the Ricci equation (2) in terms of some metric g_{abc} is the Riemann tensor of that metric when K_{abcd} $(= g_{ai}K^i_{bcd})$ is of one of the following types,

(i) *$\underset{0}{K}_{abcd}$ is of Petrov type I, II or III.*

(ii) *$\mathcal{K}_{abcd}$ is of Petrov type I, II or III.*

(iii) $\underset{0}{K}_{abcd}$ *is of Petrov type* D *and* $\mathcal{K}_{abcd}$ *is of Petrov type* N .

(iv) $\underset{0}{K}_{abcd}$ *is of Petrov type* N *and* $\mathcal{K}_{abcd}$ *is of Petrov type* D .

Of course – in view of Rendall's counterexamples [2] – it was not expected that the result would apply in all cases. However it is hoped that the above result may be strengthened a little.

4. DISCUSSION

The counter-example given by Rendall [2] was,

$$K^a_{bcd} = \alpha R^a_{bcd} \tag{11}$$

where α is constant and where R^a_{bcd} is the Riemann tensor of a metric g_{ab} of constant curvature i.e.

$$K^a_{bcd} = \alpha \frac{R}{12}(\delta^a_c g_{bd} - \delta^a_d g_{bc}) \ . \tag{12}$$

Although K^a_{bcd} satisfies the Ricci equation in the metric g_{ab} for *any* constant α , Rendall found that this tensor could not be a Riemann tensor (for any metric) except for the special case where $\alpha = 1$. When K^a_{bcd} has the form (12) it follows immediately that, with respect to the metric g_{ab} , the Weyl tensor and Plebanski tensor are both of Petrov type O ; therefore this is one of the special cases not satisfying any of the conditions of the theorem in the last section.

In Rendall's proof for the generic case – i.e. for an open dense set of curvature candidates in the Whitney C^∞ topology – the missing step was simply to show the existence of *one* curvature candidate K^a_{bcd} for which the algebraic condition (5) implies that P^a_{bcd} be identically zero. Our results – for the case where K^a_{bcd} satisfies (1) and (2) – clearly fullfill that condition, and so by Rendall's arguments a generic curvature candidate K^a_{bcd} , satisfying (1) and (2), implies that P^a_{bcd} is identically zero, and hence the curvature candidate K^a_{bcd} is equal to the Riemann tensor R^a_{bcd} of the metric g_{ab} .

However the results given in the last section also have quite specific applications. A special case of the result in Section 3 is that if one could obtain a trace-free curvature candidate of Petrov type I (II or III) together with a metric tensor that ensures that the Ricci equation (2) is satisfied, then the metric is automatically a *vacuum* metric, and the curvature candidate is automatically its Riemann tensor. This suggests the possibility of finding solutions of Einstein's vacuum equations – *without having to use them explicitly.*

It is emphasised again that the «general circumstances» stated explicitly in the result in Section 3 may possibly be made even more general. The exclusion of some cases of Petrov types N and O is not really surprising – in these cases the rank of the bivector space of K^a_{bcd} is very low; such cases are usually excluded in such results associated with algebraic constraints on the Riemann tensor [10]. However the apparent exclusion of Petrov type D is a little surprising. This question is presently being investigated, together with the possibility of whether the additional constraint of the Bianchi equations – as originally suggested by Rendall [2,3] – would affect those cases which are presently excluded.

As already pointed out, in this paper it was assumed that the covariant derivative occurring in the Ricci equation (2) was defined in terms of the same metric tensor as in the algebraic symmetry condition (1c). To answer the second part of Rendall's question we will need to consider the (presumably larger) set of curvature candidates where the metric tensor in (1c) differs from the metric tensor of the covariant derivative in (2); even more generally the covariant derivative in (2) could be defined in terms of a symmetric affine connection – which need not be metric. These more general conditions will be looked at in a subsequent paper.

However there is a related question which arises from the result of this paper; whether a curvature candidate K^a_{bcd} satisfying (1) and (2) – in terms of the same metric g_{ab} – can also be the Riemann tensor R^a_{bcd} of a *different* metric from g_{ab}. Considerable attention has been given to the question of whether two different metrics can have the same Riemann tensor, and it is now known that the Riemann tensor R^a_{bcd} determines its metric tensor g_{ab} uniquely (up to a conformal constant) – for a very large class of space-times, [10]. The class of spaces for which this uniqueness result holds is given in terms of the rank of the Riemann tensor, whereas the class of spaces for which the theorem in Section 3 holds is given in terms of the Petrov class of the Weyl and Plebanski tensors. It is obvious that there is substantial overlap between the two classes but it is not immediately obvious whether all of the spaces obeying our theorem also obey the uniqueness theorem. The detailed answer to this question will be considered within the wider context of the subsequent paper referred to above.

Finally, it is known that the Bianchi equations alone impose considerable constraints on a curvature candidate [1] and it would be expected that only a small set of additional constraint equations would be needed to characterise the Riemann tensor. The Ricci equations used in this paper are in fact a very large set, and it would seem that there should be a smaller set which would combine with the Bianchi equations to provide the same effect in a more efficient and useful way. Alternative systems are presently under investigation.

ACKNOWLEDGMENTS

I am grateful to Dr. Graham Hall for informing me of Rendall's results and to Dr. Alan Rendall for supplying preprints of his papers.

APPENDIX

We shall illustrate how these results in Section 3 were obtained.

It will be convenient to extend the N.P. notation [5] as follows: the five independent complex tetrad components of the trace-free $\underset{0}{K}{}^a_{bcd}$ will be labelled $\tilde{\Psi}_i$, the six independent components of the trace-free $\underset{0}{K}{}_{ab}$ will be labelled by $\tilde{\Phi}_{ij}$ and the scalar K will be labelled by $24\tilde{\Lambda}$ – following the same pattern as for the Riemann tensor.

Petrov type I $\underset{0}{K}{}^a_{bcd}$

In canonical form $\underset{0}{K}{}^a_{bcd}$ satisfies

(A1a) $\tilde{\Psi}_1 = 0 = \tilde{\Psi}_3$

(A1b) $\tilde{\Psi}_0 = \tilde{\Psi}_4$.

Multiplying equations (9) by $Z_{1a}Z_3^bZ_3^cZ_4^dZ_m^eZ_n^f$ gives

(A2a) $3\tilde{\Psi}_2P_{13mn} + \tilde{\Psi}_0P_{24mn} = 0$

and by $Z_{2a}Z_4^bZ_3^cZ_4^dZ_m^eZ_n^f$ gives

(A2b) $3\tilde{\Psi}_2P_{24mn} + \tilde{\Psi}_4P_{13mn} = 0$

which together give

(A3) $P_{24mn} = 0 = P_{13mn}$

for all m, n.

Multiplying equation (9) by $Z_{2a}Z_4^bZ_2^cZ_4^dZ_m^eZ_n^f$ gives

(A4) $\tilde{\Psi}_4(P_{21mn} + P_{34mn}) = 0$

for all m, n.

Equation (A3) together with equation (A4) shows

(A5) $K^a_{bcd} = R^a_{bcd}$

for all a, b, c, d.

Petrov type II and III $\underset{0}{K}{}^a_{bcd}$

When the usual canonical forms are chosen for these classes, it is easily shown by the above technique that the curvature candidate K^a_{bcd} is identical to the Riemann tensor R^a_{bcd} for such classes.

Petrov type I, II, III $\mathcal{K}^a_{bcd}$

Equation (8) which imposes constraints on the tensor P^a_{bcd} by the symmetric second order trace-free tensor K_{ab}

$$0 = \underset{0}{K}_{ic}\ P^i_{bef} + \underset{0}{K}_{ib}\ P^i_{cef} \tag{A6}$$

has exactly the same form as equation

$$x_{ic} R^i_{bef} + x_{ib} R^i_{cef} = 0 \tag{A7}$$

considered in [8]. There the constraints imposed on R^a_{bcd} by the symmetric second order tensor x_{ab} were considered for the various classes of the Plebanski tensor $\mathcal{P}_x$ (the Plebanski tensor formed from the trace-free part of x_{ab} as in equation (10)). The results in Table 1, [8] can be immediately applied to equation (A6). It therefore follows from the table that if $\mathcal{K}^a_{bcd}$ is of Petrov type I, II or III then P^a_{bcd} is identically zero and so K^a_{bcd} is identical to the Riemann tensor R^a_{bcd}.

Petrov type N $\underset{0}{K}{}^a_{bcd}$ and Petrov type D $\mathcal{K}^a_{bcd}$

When the canonical form for $\underset{0}{K}{}^a_{bcd}$ type N,

$$\tilde{\Psi}_0 = \tilde{\Psi}_1 = \tilde{\Psi}_2 = \tilde{\Psi}_3 = 0 \tag{A8}$$

is substituted into equation (5) it is found that

$$P_{13mn} = 0 = P_{12mn} + P_{34mn} \tag{A9}$$

for all m, n. This means that P^a_{bcd} is not necessarily identically zero; the components P_{2423} and P_{2424} may be non-zero i.e. $\underset{0}{P}{}^a_{bcd}$ may be Petrov type N (or O) and $\mathcal{P}^a_{bcd}$ may be Petrov type 0 .

Turning next to $\mathcal{K}^a_{bcd}$ of Petrov type D and substituting the canonical forms into equation (8) gives the result from Table 1, [8] that at least one of $\underset{0}{P}{}^a_{bcd}$, $\mathcal{P}^a_{bcd}$ must be Petrov type D (the other type D or O) -- or else P^a_{bcd} is identically zero.

When these results are taken together it follows that P^a_{bcd} must be identically zero and so K^a_{bcd} is identical to the Riemann tensor R^a_{bcd}.

Other cases

Using the above techniques the remaining results in Section 3 can be obtained.

REFERENCES

[1] S. BRIAN EDGAR: *Role of Bianchi equations in defining Riemann tensor.* Preprint, 1988.

[2] ALAN D. RENDALL: *Insufficiency of the Ricci and Bianchi identities for Characterising Curvature.* Journal of Geometry and Physics **6**, 160 (1989).

[3] ALAN D. RENDALL: *Describing space-times by their curvature.* Proceedings of the Conference on Mathematical Relativity, Canberra, 1988: ed. R. Bartnick Australian National University (1989) p. 125-136.

[4] A.Z. PETROV: *Einstein spaces.* Pergamon Press (1969).

[5] E.T. NEWMAN: and R. PENROSE: *An approach to gravitational radiation by a method of spin coefficients.* J. Math. Phys. **3**, 566-578 (1962).

[6] J. PLEBANSKI: *The algebraic structure of the tensor of matter.* Acta Phys. Pol. **26**, 963-1020 (1964).

[7] C.B.G. MCINTOSH, J. FOYSTER and A.W.C. LUN: *The classification of the Ricci and Plebanski tensors in general relativity using Newman-Penrose formalism.* J. Math. Phys. **22**, 2620-2623 (1981).

[8] C.B.G. MCINTOSH and W.D. HALFORD: *The Ricci tensor, the metric tensor and curvature collinations in general relativity.* J. Math. Phys. **23**, 436-441 (1982).

[9] D. KRAMER, et al.: *Exact solutions of Einstein's Field Equations.* Cambridge University Press (1980).

[10] G.S. HALL: *Curvature and metric in general relativity.* Classical General Relativity. eds. W.B. Bonnor et al. Cambridge University Press (1984) p. 103-120.

Manuscript received: January 1, 1990
Revised: December 22, 1990